Chapter 9 Sequence and Series
Section 9.2: Arithmetic Sequences and Partial Sums
Section Objectives: To recognize, write, and manipulate arithmetic sequences.

I. Arithmetic Sequences

- An arithmetic sequence has terms that have a common difference, $d$.
  - Recursive Formula:
    - $a_{k+1} = a_k + d$
  - Alternate Formula:
    - $a_n = (\text{common difference})(\# \text{ terms}) + (\text{first term} - d)$
    - $a_n = dn + c$, where $c = a_1 - d$
    - If you look at the explicit formula as a linear function, we can compare it to $y = mx + b$

Example 1: Is the sequence arithmetic? If so, find the difference.

a) $7, 3, -1, -5, -9, ...$
   yes, $d = -4$

b) $x, 3x, 5x, 7x ...$
   yes, $d = 2x$

- Develop the Explicit $n^{\text{th}}$ term formula:

  $$a_n = dn + c; \text{ since } c = a_1 - d$$
  $$a_n = dn + a_1 - d$$
  $$a_n = a_1 + dn - d$$
  $$a_n = a_1 + d(n - 1) \text{ or } a_n = a_1 + (n - 1)d$$

Example 2

a). Write the first 5 terms of the arithmetic sequence: $a_1 = 6$
   $a_{k+1} = a_k - 5$
   $$a_1 = 6$$
   $$a_2 = a_1 - 5 = 6 - 5 = 1$$
   $$a_3 = a_2 - 5 = 1 - 5 = -4$$
   $$a_4 = a_3 - 5 = -4 - 5 = -9$$
   $$a_5 = a_4 - 5 = -9 - 5 = -14$$

b). Find the $n^{\text{th}}$ term of the sequence.
   The constant difference is $-5$ and the first term is $6$; so this is all the information needed to write the explicit formula for the sequence.
   $$a_n = a_1 + d(n - 1)$$
   $$a_n = 6 + (-5)(n - 1)$$
   $$a_n = 6 - 5n + 5$$
   $$a_n = 11 - 5n$$
Example 3:
Find the formula for the $n$th term of the arithmetic sequence.

a) common difference $d = 2$, first term $a_1 = 7$

$$a_n = 7 + 2(n - 1) \quad \text{or} \quad a_n = 2n + (7-2)$$

$$a_n = 2n + 5$$

b) common difference $d = -5$, first term $a_1 = 9$

$$a_n = 9 + (n - 1)(-5)$$

$$a_n = 9 - 5n + 5$$

$$a_n = -5n + 14$$

Example 4:
Find the $n$th term of a sequence with fifth term 19 and ninth term 27.

$$a_5 = 19 = a_1 + (5 - 1)d = a_1 + 4d$$

$$a_9 = 27 = a_1 + (9 - 1)d = a_1 + 8d$$

$$27 = a_1 + 8d$$

$$-19 = a_1 + 4d$$

$$8 = 4d$$

$$2 = d$$

In this problem, we do not know the first term or the common difference!

We are going to use the explicit formula to create a system of 2 equations that we have to solve.

$$19 = a_1 + 8$$

$$11 = a_1$$

$$a_n = 11 + (n - 1)(2)$$

$$a_n = 2n + 9$$

Example 5:
Find the $n$th term of the sequence if $a_7 = \frac{3}{5}$ and $a_{19} = 7$.

$$a_7 = \frac{3}{5} = a_1 + d(6) \quad \Rightarrow \quad 7 = a_1 + 18d$$

$$a_{19} = 7 = a_1 + d(18) \quad \Rightarrow \quad \frac{3}{5} = a_1 + 6d$$

Subtract to find $d$:

$$\frac{32}{5} = 12d$$

$$\frac{32}{12} = d$$

Find the 1st term:

$$\frac{3}{5} = a_1 + \frac{8}{15}$$

$$-\frac{13}{5} = a_1$$

$$d = \frac{8}{15}$$

Find the formula for the $n$th term:

$$a_n = -\frac{13}{5} + \frac{8}{15}(n-1)$$

$$a_n = -\frac{47}{15} + \frac{8}{15}n$$
DAY 2
II. Sum of a Finite Arithmetic Sequence

- Let’s develop a formula for the sum of the first $n$ terms of an arithmetic sequence:
  - Let $S_n$ represent the sum of $n$ terms.

Add the following 2 equations:

\[
S_n = a_1 + a_2 + a_3 + \ldots + a_n = a_1 + (a_1 + d) + (a_1 + 2d) + (a_1 + 3d) + \ldots
\]
\[
S_n = a_n + a_{n-1} + a_{n-2} + \ldots + a_1 = a_n + (a_n - d) + (a_n - 2d) + (a_n - 3d) + \ldots
\]

\[
2S_n = (a_1 + a_n) + (a_1 + a_n) + (a_1 + a_n) + \ldots + (a_1 + a_n)
\]
\[
2S_n = n(a_1 + a_n)
\]
\[
S_n = \frac{n(a_1 + a_n)}{2}
\]

- This is the formula to find the partial sum of an arithmetic sequence with $n$ terms.

Example 4:
Evaluate $\sum_{n=1}^{20}7n+1$.

Find the first term: $a_1 = 8$
Find the last term: $a_{20} = 141$
\[
S_{20} = \frac{20(8+141)}{2} = 1490
\]

Example 5:
Find the sum of the first 20 terms of $a_n = 28 - 5n$.

Find the first term: $a_1 = 23$. Find the last (20th) term: $a_{20} = -72$
\[
S_{20} = \frac{20(23+(-72))}{2} = -490
\]

Example 6:
Find the 15th partial sum of 2, 5, 8, 11, ... 
\[
a_1 = 2, \ d = 3. \ \text{We want the sum of the first 15 terms, beginning with 2.}
\]
Using: $a_n = a_1 + (n - 1)d$
\[
a_{15} = 2 + (14)(3) = 44
\]
So we now have the first term and the 15th term.
\[
S_{15} = \frac{15(2+44)}{2} = 345
\]

Carl Gauss, as a 10-year old boy in 1787, figured how out to add the numbers from 1 to 100 by using this sum formula all by himself. It is even said that he learned numbers and arithmetic operations before he learned to talk!
Example 7:
Each row of an auditorium has 2 more seats than the preceding row. Find the seating capacity of the
auditorium if the front row has 30 seats and there are 40 rows.

The first row: \(a_1 = 30\),
Each row has 2 more, so the common difference, \(d = 2\)
The last (40th) row: \(a_{40} = 30 + (39)(2) = 108\)

Therefore, using the formula for partial sums:
\[S_{40} = \frac{40(30 + 108)}{2} = 2760 \text{ seats}\]

Example 8:
Suppose you put $100 away at the end of the month. You put away money each month and increase the
amount by $5 each time. How much money will you have after 1 year?

The first year, you have $100: \(a_1 = 100, d = 5\)
In the 12th month (1 year), you will save: \(a_{12} = 100 + (11)(5) = 155\)
To find how much money you have, you have to total up all the monies that you have accumulated during
those 12 months: \(S_{12} = \frac{12(100 + 155)}{2} = \$1530\)

We are also able to calculate different partial sums such as these: \(\sum_{n=11}^{50} 3n\).
\[
\sum_{n=11}^{50} 3n = 3(11) + 3(12) + \ldots + 3(50)
\]
\[
\sum_{n=11}^{50} 3n = \sum_{n=1}^{50} 3n - \sum_{n=1}^{10} 3n
\]
\[
= [3(1) + 3(2) + \ldots + 3(50)] - [3(1) + 3(2) + \ldots + 3(9) + 3(10)]
\]
\[
= 3(11) + 3(12) + \ldots + 3(50)
\]
\[a_1 = 3(1) = 3, \quad a_{10} = 3(10) = 30, \quad a_{50} = 3(50) = 150\]

There are a couple of different ways to solve this problem. The
first is to realize that if we want to sum up the sequence \(3n\) from
11 to 50, it’s the same as finding the sum from 1 to 50 and
subtracting the ones from 1 to 10.

(i) \[
\sum_{n=1}^{50} 3n - \sum_{n=1}^{10} 3n = \frac{50(3 + 150)}{2} - \frac{10(3 + 30)}{2}
\]
\[
= 25(153) - 5(33)
\]
\[
= 3825 + 165
\]
\[
= 3660
\]

(ii) \[
\sum_{n=11}^{50} 3n = \frac{40(33 + 150)}{2}
\]
\[
= 20(183)
\]
\[
= 3660
\]

Here, the formula is used a little bit differently.
\(n\) refers to the total number of terms being summed up (in this case from 11 to 50 inclusive, there are
40 terms). And the first term is the first term considered by the series by using the lower limit. The last
term is the one by using the upper limit.