Formulas and Theorems for Reference

I. Trigonometric Formulas

Pythagorean Identities

1. $\sin^2\theta+\cos^2\theta=1$
2. $1+\tan^2\theta=\sec^2\theta$
3. $\cot^2\theta+1=\csc^2\theta$

Even/Odd Identities

4. $\sin(-\theta)=-\sin\theta$
5. $\cos(-\theta)=\cos\theta$
6. $\tan(-\theta)=-\tan\theta$

Sum/Difference Formulas

7. $\sin(\alpha\pm\beta)=\sin\alpha\cos\beta\pm\sin\beta\cos\alpha$
8. $\cos(\alpha\pm\beta)=\cos\alpha\cos\beta\mp\sin\alpha\sin\beta$

Double Angle Formulas

9. $\sin 2\theta=2\sin\theta\cos\theta$
10. $\cos 2\theta=\begin{cases}\cos^2\theta-\sin^2\theta\\ 2\cos^2\theta-1\\ 1-2\sin^2\theta\end{cases}$

Power-Reducing Formulas

11. $\cos^2\theta=\frac{1}{2}(1+\cos 2\theta)$
12. $\sin^2\theta=\frac{1}{2}(1-\cos 2\theta)$

Quotient and Reciprocal Identities

13. $\tan\theta=\frac{\sin\theta}{\cos\theta}=\frac{1}{\cot\theta}$
14. $\cot\theta=\frac{\cos\theta}{\sin\theta}=\frac{1}{\tan\theta}$
15. $\sec\theta=\frac{1}{\cos\theta}$
16. $\csc\theta=\frac{1}{\sin\theta}$

Co-Function Identities

17. $\cos\left(\frac{\pi}{2}-\theta\right)=\sin\theta$
18. $\sin\left(\frac{\pi}{2}-\theta\right)=\cos\theta$

II. Differentiation Formulas

1. $\frac{d}{dx}\left(x^n\right)=nx^{n-1}$

2. $\frac{d}{dx}\left(fg\right)=fg'+gf'$

3. $\frac{d}{dx}\left(\frac{f}{g}\right)=\frac{gf'-fg'}{g^2}$

4. $\frac{d}{dx}f\left(g(x)\right)=f'\left(g(x)\right)g'(x)$

5. $\frac{d}{dx}\left(\sin x\right)=\cos x$

6. $\frac{d}{dx}\left(\cos x\right)=-\sin x$

7. $\frac{d}{dx}\left(\tan x\right)=\sec^2 x$

8. $\frac{d}{dx}\left(\csc x\right)=-\csc x\cot x$

9. $\frac{d}{dx}\left(\sec x\right)=\sec x\tan x$

10. $\frac{d}{dx}\left(\cot x\right)=-\csc^2 x$

11. $\frac{d}{dx}\left(e^x\right)=e^x$

12. $\frac{d}{dx}\left(a^x\right)=a^x\ln a$

13. $\frac{d}{dx}\left(\ln x\right)=\frac{1}{x}$

14. $\frac{d}{dx}\left(\tan^{-1} x\right)=\frac{1}{1+x^2}$

15. $\frac{d}{dx}\left(\sin^{-1} x\right)=\frac{1}{\sqrt{1-x^2}}$

16. $\frac{d}{dx}\left(\sec^{-1} x\right)=\frac{1}{|x|\sqrt{x^2-1}}$

III. Integration Formulas

1. $\int a\,dx = ax + C$

2. $\int x^n dx = \dfrac{x^{n+1}}{n+1} + C,\ n \neq 1$

3. $\int \dfrac{1}{x} dx = \ln|x| + C$

4. $\int e^x dx = e^x + C$

5. $\int a^x dx = \dfrac{a^x}{\ln a} + C$

6. $\int \ln x\,dx = x\ln x - x + C$

7. $\int \sin x\,dx = -\cos x + C$

8. $\int \cos x\,dx = \sin x + C$

9. $\int \tan x\,dx = \ln|\sec x| + C$ or $-\ln|\cos x| + C$

10. $\int \cot x\,dx = \ln|\sin x| + C$

11. $\int \sec x\,dx = \ln|\sec x + \tan x| + C$

12. $\int \csc x\,dx = \ln|\csc x - \cot x| + C$

13. $\int \sec^2 x\,dx = \tan x + C$

14. $\int \sec x \tan x = \sec x + C$

15. $\int \csc^2 x\,dx = -\cot x + C$

16. $\int \csc x \cot x\,dx = -\csc x + C$

17. $\int \tan^2 x\,dx = \tan x - x + C$

18. $\int \dfrac{dx}{a^2 + x^2} = \dfrac{1}{a}\tan^{-1}\left(\dfrac{x}{a}\right) + C$

19. $\int \dfrac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\dfrac{x}{a}\right) + C$

20. $\int \dfrac{dx}{x\sqrt{x^2 - a^2}} = \dfrac{1}{a}\sec^{-1}\dfrac{|x|}{a} + C$

IV. Formulas and Theorems

1. Limits and Continuity

A function $y = f(x)$ is continuous at $x = a$ if:

i) $f(a)$ is defined (exists)

ii) $\lim_{x \to a} f(x)$ exists, and

iii) $\lim_{x \to a} f(x) = f(a)$

Otherwise, f is discontinuous at $x = a$.

The limit $\lim_{x \to a} f(x)$ exists if and only if both corresponding one-sided limits exist and are equal – that is, $\lim_{x \to a} f(x) = L \Leftrightarrow \lim_{x \to a^+} f(x) = L = \lim_{x \to a^-} f(x)$

2. Intermediate Value Theorem

A function $y = f(x)$ that is continuous on a closed interval $[a,b]$ takes on every value between $f(a)$ and $f(b)$.

Note: If f is continuous on $[a,b]$ and $f(a)$ and $f(b)$ differ in sign, then the equation $f(x) = 0$ has at least one solution in the open interval (a,b).

3. Limits of Rational Functions as $x \to \pm\infty$

1. $\lim_{x \to \pm\infty} \frac{f(x)}{g(x)} = 0$ if the degree of $f(x) <$ the degree of $g(x)$

   Example: $\lim_{x \to \infty} \frac{x^2 - 2x}{x^3 + 3} = 0$

2. $\lim_{x \to \pm\infty} \frac{f(x)}{g(x)}$ is infinite if the degree of $f(x) >$ the degree of $g(x)$

   Example: $\lim_{x \to +\infty} \frac{x^3 + 2x}{x^2 + 8} = \infty$

3. $\lim_{x \to \pm\infty} \frac{f(x)}{g(x)}$ is finite if the degree of $f(x) =$ the degree of $g(x)$

   Note: The limit will be the ratio of the leading coefficient of $f(x)$ to $g(x)$

   Example: $\lim_{x \to \infty} \frac{2x^2 - 3x + 2}{10x - 5x^2} = -\frac{2}{5}$

4. Horizontal and Vertical Asymptotes

1. A line $y = b$ is a horizontal asymptote of the graph of $y = f(x)$ if either $\lim_{x \to \infty} f(x) = b$ or $\lim_{x \to -\infty} f(x) = b$.

2. A line $x = a$ is a vertical asymptote of the graph of $y = f(x)$ if either $\lim_{x \to a^+} f(x) = \pm\infty$ or $\lim_{x \to a^-} f(x) = \pm\infty$

5. Average and Instantaneous Rate of Change

1. Average Rate of Change: If (x_0, y_0) and (x_1, y_1) are points on the graph of $y = f(x)$, then the average rate of change of y with respect to x over the interval $[x_0, x_1]$ is $\frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{y_1 - y_0}{x_1 - x_0} = \frac{\Delta y}{\Delta x}$

2. Instantaneous Rate of Change: If (x_0, y_0) is a point on the graph of $y = f(x)$, then the instantaneous rate of change of y with respect to x at x_0 is $f'(x_0)$.

6. Definition of Derivative

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \text{ or } f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

The latter definition of the derivative is the instantaneous rate of change of $f(x)$ with respect to x at $x = a$.

Geometrically, the derivative of a function at a point is the slope of the tangent line to the graph of the function at that point.

7. The number e as a limit

   1. $\lim_{n \to +\infty} \left(1 + \frac{1}{n}\right)^n = e$
   2. $\lim_{n \to 0} (1+n)^{\frac{1}{n}} = e$

8. Rolle's Theorem

If f is continuous on $[a,b]$ and differentiable on (a,b) such that $f(a) = f(b)$, then there is at least one number c in the open interval (a,b) such that $f'(c) = 0$.

9. Mean Value Theorem

If f is continuous on $[a,b]$ and differentiable on (a,b), then there is at least one number c in the open interval (a,b) such that $\frac{f(b) - f(a)}{b - a} = f'(c)$.

10. Extreme Value Theorem

If f is continuous on the closed interval $[a,b]$, then $f(x)$ has both a maximum and a minimum on $[a,b]$.

11. To find the maximum and minimum values on a function $y = f(x)$, locate

   1. the points where $f'(x)$ is zero or where $f'(x)$ fails to exist
   2. the end points, if any, on the domain of $f(x)$.

   Note: These are the only candidates for the value of x where $f(x)$ may have a maximum or a minimum.

12. Let f be differentiable for $a < x < b$ and continuous for $a \le x \le b$.

   1. f is increasing on $[a,b]$ if and only if $f'(x) > 0$ for every x in (a,b). $f'(x) < 0$ for every x in (a,b)
   2. f is decreasing on $[a,b]$ if and only if .

13. Suppose that $f''(x)$ exists on the interval (a,b).

   1. f is concave upward on (a,b) if and only if $f''(x) > 0$ for every x in (a,b).
   2. f is concave downward on (a,b) if and only if $f''(x) < 0$ for every x in (a,b).

To locate the points of inflection of $y=f(x)$, find the points where $f''(x)=0$ OR where $f''(x)$ fails to exist. These are the only candidates where $f(x)$ may have a point of inflection. Then test these points to make sure that $f''(x)<0$ on one side and $f''(x)>0$ on the other.

14. If a function is differentiable at a point $x = a$, it is continuous at that point. The converse is false, i.e. continuity does not imply differentiability.

15. Local Linearity and Linear Approximation

    The linear approximation to $f(x)$ near $x=x_0$ is given by $y=f(x_0)+f'(x_0)(x-x_0)$ for x sufficiently close to x_0.

    To estimate the slope of a graph at a point – just draw a tangent line to the graph at that point. Another way is (by using a graphing calculator) to "zoom in" around the point in question until the graph "looks" straight. This method will always work. If we "zoom in" and the graph looks straight at a point, say $(a,f(a))$, then the function is locally linear at that point.

    The graph of $y=|x|$ has a sharp corner at $x = 0$. This corner cannot be smoothed out by "zooming in" repeatedly. Consequently, the derivative of $|x|$ does not exist at $x = 0$, hence, is not locally linear at $x = 0$.

16. Newton's Method

    Let f be a differentiable function and suppose r is a real zero of f. If x_n is an approximation to r, then the next approximation x_{n+1} is given by $x_{n+1}=x_n-\dfrac{f(x_n)}{f'(x_n)}$ provided $f'(x_n)\neq 0$. Successive approximations can be found using this method.

17. Dominance and Comparison of Rates of Change

    Logarithmic functions grow slower than any power function (x^n).

    Among power functions, those with higher powers grow faster than those with lower powers.
    All power functions grow slower than any exponential function $(a^x, a>1)$.

    Among exponential functions, those with larger bases grow faster than those with smaller bases.
    We say, that as $x\to\infty$:

    1. $f(x)$ grows faster than $g(x)$ if $\lim\limits_{x\to\infty}\dfrac{f(x)}{g(x)}=\infty$ or $\lim\limits_{x\to\infty}\dfrac{g(x)}{f(x)}=0$.

       If $f(x)$ grows faster than $g(x)$ as $x\to\infty$, then $g(x)$ grows slower than $f(x)$ as $x\to\infty$.

    2. $f(x)$ and $g(x)$ grow at the same rate as $x\to\infty$ if $\lim\limits_{x\to\infty}\dfrac{f(x)}{g(x)}=L\neq 0$ (L is finite and nonzero).

    For example,

    1. e^x grows faster than x^3 as $x\to\infty$ since $\lim\limits_{x\to\infty}\dfrac{e^x}{x^3}=\infty$
    2. x^4 grows faster than $\ln x$ as $x\to\infty$ since $\lim\limits_{x\to\infty}\dfrac{x^4}{\ln x}=\infty$
    3. x^2+2x grows at the same rate as x^2 as $x\to\infty$ since $\lim\limits_{x\to\infty}\dfrac{x^2+2x}{x^2}=1$

To find some of these limits as $x\to\infty$, you may use a graphing calculator. Make sure that an appropriate window is used.

18. Inverse Functions

1. If f and g are two functions such that $f\left(g(x)\right)=x$ for every x in the domain of g, and, $g\left(f(x)\right)=x$, for every x in the domain of f, then, f and g are inverse functions of each other.
2. A function f has an inverse function if and only if no horizontal like intersects its graph more than once.
3. If f is either increasing or decreasing in an interval, then f has an inverse function over that interval.
4. If f is differentiable at every point on an interval I, and $f'(x)\neq 0$ on I, then $g=f^{-1}(x)$ is differentiable at every point of the interior of the interval $f(I)$ and $g'\left(f(x)\right)=\dfrac{1}{f'(x)}$.

19. Properties of $y=e^x$

1. The exponential function $y=e^x$ is the inverse function of $y = \ln x$.
2. The domain is the set of all real numbers, $-\infty<x<\infty$.
3. The range is the set of all positive numbers, $y > 0$.
4. $\dfrac{d}{dx}\left(e^x\right)=e^x$.
5. $e^{x_1}\cdot e^{x_2}=e^{x_1+x_2}$.
6. $y=e^x$ is continuous, increasing, and concave up for all x.
7. $\lim\limits_{x\to+\infty} e^x=+\infty$ and $\lim\limits_{x\to-\infty} e^x=0$
8. $e^{\ln x}=x$, for $x>0$; $\ln\left(e^x\right)$ for all x.

20. Properties of $y=\ln x$

1. The natural logarithm function $y=\ln x$ is the inverse of the exponential function $y=e^x$.
2. The domain of $y=\ln x$ is the set of all positive numbers, $x > 0$.
3. The range of $y=\ln x$ is the set of all real numbers, $-\infty<y<\infty$.
4. $y=\ln x$ is continuous and increasing everywhere on its domain.
5. $\ln(ab)=\ln a+\ln b$.
6. $\ln\left(\dfrac{a}{b}\right)=\ln a-\ln b$.
7. $\ln a^r=r\ln a$.
8. $y=\ln x < 0$ if $0 < x < 1$.
9. $\lim\limits_{x\to+\infty}\ln x=+\infty$ and $\lim\limits_{x\to0^+}\ln x=-\infty$.
10. $\log_a x=\dfrac{\ln x}{\ln a}$.

21. L'Hopital's Rule

If $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)}$ is of the form $\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$, and if $\lim\limits_{x\to a}\dfrac{f'(x)}{g'(x)}$ exists, then $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)}=\lim\limits_{x\to a}\dfrac{f'(x)}{g'(x)}$.

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22. Trapezoidal Rule

If a function f is continuous on the closed interval [a, b], where [a, b] has been partitioned into n sub-intervals $[x_0,x_1],[x_1,x_2],\ldots,[x_{n-1},x_n]$, each length of $\dfrac{b-a}{n}$, then

$$\int_a^b f(x)dx\approx\frac{b-a}{n}\left[f(x_0)+2f(x_1)+2f(x_2)+\ldots+2f(x_{n-1})+f(x_n)\right].$$

The Trapezoidal Rule is the average of the Left and Right hand sums.

23. Simpson's Rule

Let f be continuous on [a, b].

Then $\int_a^b f(x)dx \approx \frac{b-a}{3n}\left[f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+...+4f(x_{n-1})+f(x_n)\right]$,

where n is an even number of subintervals of equal length on [a, b].

24. Properties of the Definite Integral

Let $f(x)$ and $g(x)$ be continuous on $[a,b]$.

1. $\int_a^b c\cdot f(x)dx = c\int_a^b f(x)dx$ for any constant x.
2. $\int_a^a f(x)dx = 0$
3. $\int_b^a f(x)dx = -\int_a^b f(x)dx$
4. $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$, where f is continuous on an interval containing the numbers a, b, and c, regardless of the order of a, b, and c.
5. If $f(x)$ is an odd function, then $\int_{-a}^a f(x)dx = 0$
6. If $f(x)$ is an even function, then $\int_{-a}^a f(x)dx = 2\int_0^a f(x)dx$
7. If $f(x)\geq 0$ on $[a,b]$, then $\int_a^b f(x)dx \geq 0$
8. If $g(x)\geq f(x)$ on $[a,b]$, then $\int_a^b g(x)dx \geq \int_a^b f(x)dx$

25. Definition of Definite Integral as the Limit of a Sum

Suppose that a function $f(x)$ is continuous on the closed interval [a, b]. Divide the interval into n equal subintervals, of length $\Delta x = \frac{b-a}{n}$. Choose one number in each subinterval, i.e. x_1 in the first, x_2 in the second, …, x_k in the kth, …, and x_n in the nth. Then $\lim_{n\to\infty}\sum_{k=1}^{n} f(x_k)\Delta x = \int_a^b f(x)dx = F(b)-F(a)$.

26. Fundamental Theorem of Calculus

$\int_a^b f(x)dx = F(b)-F(a)$, where $F'(x)=f(x)$, or $\frac{d}{dx}\int_a^x f(x)dx = f(x)$.

27. Velocity, Speed, and Acceleration

1. The velocity of an object tells how fast it is going and in which direction. Velocity is an instantaneous rate of change.
2. The speed of an object is the absolute value of the velocity, $|v(t)|$. It tells how fast it is going disregarding its direction.
   The speed of a particle increases (speeds up) when the velocity and acceleration have the same signs. The speed decreases (slows down) when the velocity and acceleration have opposite signs.
3. The acceleration is the instantaneous rate of change of velocity – it is the derivative of the velocity – that is, $a(t)=v'(t)$. Negative acceleration (deceleration) means that the velocity is decreasing. The acceleration gives the rate at which the velocity is changing.

Therefore, it x is the displacement of a moving object and t is time, then:

i). velocity = $v(t) = x'(t) = \frac{dx}{dt}$

ii). acceleration = $a(t) = x''(t) = v'(t) = \frac{dv}{dt} = \frac{d^2x}{dt^2}$

iii). $v(t) = \int a(t)dt$

iv). $x(t) = \int v(t)dt$

Note: The average velocity of a particle over the time interval from t_0 to another time t, is

Average Velocity = $\frac{\text{Change in position}}{\text{Length of time}} = \frac{s(t) - s(t_0)}{t - t_0}$, where $a(t)$ is the position of the particle at time t.

28. The average value of $f(x)$ on $[a,b]$ is $\frac{1}{b-a}\int_a^b f(x)dx$

29. Area Between Curves

If f and g are continuous functions such that $f(x) \geq g(x)$ on $[a,b]$, then the area between the curves is $\int_a^b [f(x) - g(x)]dx$.

30. Integration By Parts

If $u = f(x)$ and $v = g(x)$ and if $f'(x)$ and $g'(x)$ are continuous, then $\int u dv = uv - \int v du$.

Note: The goal of the procedure is to choose u and dv so that $\int v du$ is easier to integrate than the original problem.

Suggestion:

When "choosing" u, remember L.I.A.T.E, where L is the logarithmic function, I is an inverse trigonometric function, A is an algebraic function, T is a trigonometric function, and E is the exponential function. Just choose u as the first expression in L.I.A.T.E (and dv will be the remaining part of the integrand.) For example, when integrating $\int x \ln x dx$, choose $u = x$, since x is an algebraic function, and A comes before E in L.I.A.T.E., and $dv = e^x dx$. One more example, when integrating $\int x \tan^{-1} x dx$, let $u = \tan^{-1} x$, since I comes before A in L.I.A.T.E., and $dv = x dx$.

31. Volumes of Solids of Revolution

Let f be nonnegative and continuous on $[a,b]$, and let R be the region bounded above by $y = f(x)$, below by the x – axis, and the sides by the lines $x = a$ and $x = b$.

1. When this region R is revolved about the x – axis, it generates a solid (having circular cross sections) whose volume $V = \int_a^b \pi [f(x)]^2 dx$.
2. When R is revolved about the y – axis, it generates a solid whose volume $V = \int_a^b 2\pi r f(x) dx$.

32. Volumes of Solids with Known Cross Sections

1. For cross sections of area $A(x)$, taken perpendicular to the x – axis, volume $= \int_a^b A(x)dx$.
2. For cross sections of area $A(y)$, taken perpendicular to the y – axis, volume $= \int_c^d A(y)dy$.

33. Trigonometric Substitution

1. For integrals involving $\sqrt{a^2-u^2}$, let $u=a\sin\theta$. Then $\sqrt{a^2-u^2}=a\cos\theta$ where $-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$.
2. For integrals involving $\sqrt{a^2+u^2}$, let $u=a\tan\theta$. Then $\sqrt{a^2+u^2}=a\sec\theta$ where $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$.

Or integrals involving $\sqrt{u^2-a^2}$, let $u=a\sec\theta$. Then $\sqrt{u^2-a^2}=\pm a\tan\theta$ where $0\le\theta<\frac{\pi}{2}$ or $\frac{\pi}{2}<\theta\le\pi$.

Use the positive value if $u > a$; the negative value if $u < -a$.

34. <u>Solving Differential Equations: Graphically and Numerically</u>
    1. <u>Slope Fields</u>
       At every point (x,y) a differential equation of the form $\frac{dy}{dx}=f(x,y)$ gives the slope of the member of family of solutions that contains the point. A slope field is a graphical representation of this family of curves. At each point in the plane, a short segment is drawn whose slope is equal to the value of the derivative at that point. These segments are tangent to the solution's graph at the point.
       The slope field allows you to sketch the graph of the solution curve even though you do not have its equation. This is done by starting at any point (usually the point given by the initial condition), and moving from one point to the next in the direction indicated by the segments of the slope field. Some calculators have built in operations for drawing slope fields; for calculators without this feature, there are programs available for drawing them.

    2. <u>Euler's Method</u>
       Euler's Method is a way of approximating points on the solution of a differential equation $\frac{dy}{dx}=f(x,y)$. The calculation uses the tangent line approximation to move from point to the next. That is, starting with the given point (x_1,y_1) – the initial condition, the point $(x_1+\Delta x, y_1+f'(x_1,y_1)\Delta x)$ approximates a nearby point to calculate a third point and so on. The accuracy of this method decreases with larger values of Δx. The error increases as each successive point is used to find the next. Calculator programs are available for doing this calculation.

35. <u>Definition of Arc Length</u>
    If the function given by $y=f(x)$ represents a smooth curve on the interval $[a,b]$, then the arc length of f between a and b is given by $s=\int_a^b\sqrt{1+|f'(x)|^2}\,dx$.

36. <u>Work</u>
    1. If an object is moved a distance D in the direction of an applied <u>constant</u> force F, then the work W done by the force is defined as $W = FD$.
    2. If an object is moved along a straight line by a continuously <u>varying</u> force $F(x)$, then the work W done by the force as the object is moved from $x = a$ to $x = b$ is given by $W=\int_a^b F(x)dx$.
    3. Hooke's Law says that the amount of force F is takes to stretch or compress a spring x units from its natural length is proportional to x. That is, $F = kx$, where k is the spring constant measured in force units per unit length.

37. <u>Improper Integral</u>
    $\int_a^b f(x)dx$ is an improper integral if
    1. f becomes infinite at one or more points of the interval of integration, or
    2. one or both of the limits of integration is infinite, or
    3. both (1) and (2) hold.

38. Parametric Form of the Derivative

If a smooth curve C is given by the parametric equations $x=f(t)$ and $y=g(t)$, then the slope of the curve C at (x,y) is $\frac{dy}{dx}=\frac{dy}{dt}\div\frac{dx}{dt}, \frac{dx}{dt}\neq 0$.

Note: The second derivative, $\frac{d^2y}{dx^2}=\frac{d}{dx}\left[\frac{dy}{dx}\right]=\frac{d}{dt}\left[\frac{dy}{dx}\right]\div\frac{dx}{dt}$.

39. Arc Length in Parametric Form

If a smooth curve C is given by $x=f(t)$ and $y=g(t)$ and these functions have continuous first derivatives with respect to t for $a\leq t\leq b$, and if the point $P(x,y)$ traces the curve exactly once as t moves from $t = a$ to $t = b$, then the length of the curve is given by $s=\int_a^b\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt=\int_a^b\sqrt{\left[f'(t)\right]^2+\left[g'(t)\right]^2}\,dt$.

40. Polar Coordinates

1. Cartesian *vs.* Polar Coordinates: The polar coordinates (r,θ) are related to the Cartesian coordinates (x,y) as follows:

   $x=r\cos\theta$ and $y=r\sin\theta$

   $\tan\theta=\frac{y}{x}$ and $x^2+y^2=r^2$

2. Area in Polar Coordinates: If f is continuous and nonnegative on the interval $[\alpha,\beta]$, then the area of the region bounded by the graph of $r=f(\theta)$ between the radial lines $\theta=\alpha$ and $\theta=\beta$ is given by $A=\frac{1}{2}\int_\alpha^\beta\left[f(\theta)\right]^2d\theta=\frac{1}{2}\int_\alpha^\beta r^2d\theta$.

41. Sequences and Series

1. If a sequence $\{a_n\}$ has a limit L, that is $\lim_{n\to\infty}a_n=L$, then the sequence is said to converge to L. If there is no limit, the series diverges. If the sequence $\{a_n\}$ converges, then its limit is unique. Keep in mind that $\lim_{n\to\infty}\frac{\ln n}{n}=0$; $\lim_{n\to\infty}x^{\frac{1}{n}}=1$; $\lim_{n\to\infty}\sqrt[n]{n}=1$; $\lim_{n\to\infty}\frac{x^n}{n!}=0$. These limits are useful and arise frequently.

2. The harmonic series $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges; the geometric series $\sum_{n=0}^{\infty}ar^n$ converges to $\frac{a}{1-r}$ if $|r|<1$ and diverges if $|r|\geq 1$ and $a\neq 0$.

3. The $p-$series $\sum_{n=1}^{\infty}\frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$.

4. Limit Comparison Test: Let $\sum_{n=1}^{\infty}a_n$ and $\sum_{n=1}^{\infty}b_n$ be a series of nonnegative terms, with $a_n\neq 0$ for all sufficiently large n, and suppose that $\lim_{n\to\infty}\frac{b_n}{a_n}=c>0$. Then the two series either both converge or both diverge.

5. Alternating Series Test: Let $\sum_{n=1}^{\infty}a_n$ be a series such that

i) the series is alternating,

ii) $|a_{n+1}| \le |a_n|$ for all n, and

iii) $\lim_{n\to\infty} a_n = 0$

Then the series converges.

6. The n th term test for Divergence: If $\lim_{n\to\infty} a_n \ne 0$, then the series diverges.

   Note: The converse if *false*, that is, if $\lim_{n\to\infty} a_n = 0$, then the series may or may not converge.

7. A series $\sum a_n$ is absolutely convergent if the series $\sum |a_n|$ converges. If $\sum a_n$ converges, but $\sum |a_n|$ does not converge, then the series is conditionally convergent. Keep in mind that if $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

8. Direct Comparison Test: If $0 \le a_n \le b_n$ for all sufficiently large n, and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges. If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.

9. Integral Test: Let $f(x)$ be a positive, continuous, and decreasing function on $[1,\infty)$ and $a_n = f(n)$. The series $\sum_{n=1}^{\infty} a_n$ will converge if the improper integral, $\int_1^{\infty} f(x)dx$ converges. If the improper integral $\int_1^{\infty} f(x)dx$ diverges, then the infinite series $\sum_{n=1}^{\infty} a_n$ diverges.

10. Ratio Test: Let $\sum a_n$ be a series with nonzero terms.

    i) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$ then the series converges absolutely.

    ii) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| > 1$, then the series is divergent.

    iii) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$, then the test is inconclusive (and another test must be used).

11. Power Series: A power series is of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \ldots + c_n x^n + \ldots \text{ or}$$

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \ldots + c_n(x-a)^n + \ldots$$

in which the center a and the coefficients $c_0, c_1, c_2, \ldots, c_n, \ldots$ are constants.
The set of all numbers x for which the power series converges is called the interval of convergence.

12. Taylor Series: Let f be a function with derivatives of all orders throughout some interval containing a as an interior point. Then the Taylor series generated by f at a is

$$\sum_{k=0}^{\infty}\frac{f^{(k)}(a)}{a!}(x-a)^k = f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\ldots\frac{f^{(n)}(a)}{n!}(x-a)^n+\ldots$$

The remaining terms after the term containing the n th derivative can be expressed as a remainder to Taylor's Theorem:

$$f(x)=f(a)+\sum_{k=1}^{\infty}f^{(n)}(a)(x-a)^n+R_n(x)=\frac{1}{n!}\int_a^x(x-t)^n f^{(n+1)}(t)dt$$

$$R_n(x)=\frac{f^{(n+1)}(c)(x-a)^{n+1}}{(n+1)!}$$

Lagrange's Form of the Remainder: , where c lies between x and a .
The series will converge for all values of x for which the remainder approaches zero as $x\to\infty$.

13. Frequently Used Series and Their Interval of Convergence

- $\frac{1}{1-x}=1+x+x^2+x^3+\cdots+x^n+\cdots=\sum_{n=0}^{\infty}x^n$; IOC $(-1,1)$
- $\frac{1}{1+x}=1-x+x^2-x^3+\cdots+(-x)^n+\cdots=\sum_{n=0}^{\infty}(-1)^n x^n$; IOC $(-1,1)$
- $e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots+\frac{x^n}{n!}+\cdots=\sum_{n=0}^{\infty}\frac{x^n}{n!}$; IOC $(-\infty,\infty)$
- $\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots+(-1)^n\frac{x^{2n+1}}{(2n+1)!}+\cdots=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}$; IOC $(-\infty,\infty)$
- $\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots+(-1)^n\frac{x^{2n}}{(2n)!}+\cdots=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}$; IOC $(-\infty,\infty)$
- $\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots+(-1)^{n-1}\frac{x^n}{n}+\cdots=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^n}{n}, -1<x\le 1$
- $\tan^{-1}x=x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\cdots+(-1)^n\frac{x^{2n+1}}{2n+1}+\cdots=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2n+1}$; IOC $[-1,1]$